

# Cosmological models and centre manifold theory

Alan D. Rendall

Max-Planck-Institut für Gravitationsphysik  
Am Mühlenberg 1, 14476 Golm, Germany

## Abstract

Centre manifold theory is applied to some dynamical systems arising from spatially homogeneous cosmological models. Detailed information is obtained concerning the late-time behaviour of solutions of the Einstein equations of Bianchi type III with collisionless matter. In addition some statements in the literature on solutions of the Einstein equations coupled to a massive scalar field are proved rigorously.

## 1 Introduction

Over the years a great deal of effort has been put into the study of homogeneous solutions of the Einstein equations coupled to various matter sources. This class of solutions has the advantage that the evolution equations reduce to ordinary differential equations. This allows a mathematically rigorous treatment of many issues for which corresponding results seem out of reach at present when partial differential equations are involved. The area of mathematics which is relevant for the study of the time evolution of homogeneous cosmological models is the theory of dynamical systems. An excellent source of background information on the application of dynamical systems to cosmology is the book edited by Wainwright and Ellis [13].

When a dynamical system (i.e. a system of ordinary differential equations) constituting a model of an aspect of the real world has been derived a natural task is to understand the qualitative behaviour of general solutions of the equations. Doing this may involve combining many techniques. In the following we concentrate on one particular type of technique. In understanding a dynamical system it is useful to determine the stationary solutions. Once these have been found it is desirable to know the behaviour of solutions which remain in a neighbourhood of one of these stationary solutions. If the system is linearized about the stationary point then there is often a relationship between solutions of the linearized system and solutions of the original system which remain near the stationary point. In the case of hyperbolic stationary points (defined below) this works very well. When a stationary point is not hyperbolic things are not so easy. In the latter case an important role is played by centre manifold theory. This theory will be illustrated by various cosmological examples in what follows.

One example of an application of centre manifold theory to homogeneous cosmological models in the literature can be found in [6]. There it was used to determine the asymptotic behaviour of spacetimes of Bianchi type VII<sub>0</sub> with a radiation fluid. A feature of this example which is typical is that it represents a degenerate or borderline case. The case of a perfect fluid with equation of state  $p = (\gamma - 1)\rho$  and  $\gamma \neq 4/3$  can be analysed without centre manifold theory [14] and it is found that the behaviour changes qualitatively when passing through the case of the radiation fluid ( $\gamma = 4/3$ ).

In this paper two different applications of centre manifold theory are considered. The first is to homogeneous spacetimes with collisionless matter. More specifically, it concerns locally rotationally symmetric (LRS) models of Bianchi type III. These play the role of a degenerate case within more general classes of Bianchi models with collisionless matter studied in [9] and [10]. The centre manifold analyses of the present paper extend and complete the results of [9] and [10]. The other application is to homogeneous and isotropic spacetimes with a massive scalar field. The centre manifold analysis provides a rigorous confirmation of heuristic conclusions on inflation in this class of spacetimes due to Belinskii et. al. [2].

The results on models with collisionless matter obtained in the following address a question raised in [10]. Suppose we have an expanding cosmological model with collisionless matter described by the Vlasov equation. The particles composing the matter have a certain non-zero velocity dispersion. It might be expected that if these particles are massive the velocity dispersion will decay with time as a consequence of the cosmological expansion so that the spacetime will look more and more like a dust model in the phase of unlimited expansion. It was shown in [10] that this is indeed the case for LRS models of Bianchi types I and II. For Bianchi type III, however, despite the amount of information on the dynamics obtained in [10], this issue was not settled. The reason for the difficulty is that in Bianchi type III one of the scale factors has an anomalously slow rate of expansion. In fact the results of [10] did not even suffice to show that this scale factor is unbounded. It is shown below that dust-like asymptotics also occurs in the case of Bianchi type III and that the problematic scale factor does increase without limit. An analogous result for massless particles is also obtained, thus completing the results of [9]. It is to be hoped that these results will also open the way to further progress in understanding inhomogeneous models with collisionless matter such as those discussed in [1].

The second section contains some general information about centre manifold theory. Some equations for spacetimes of Bianchi type III with kinetic matter are recalled in Section 3. Centre manifold theory is applied to Bianchi III spacetimes with collisionless matter in the case of massive particles in Section 4 and in the case of massless particles in Section 5. Section 6 contains results on homogeneous and isotropic solutions with a massive scalar field. The final section presents some insights which can be obtained from the results in the body of the paper.

## 2 Centre manifold theory

Let  $\dot{x} = f(x)$  be a system of ordinary differential equations, where  $x$  is a point of  $\mathbf{R}^n$  and the function  $f$  is smooth ( $C^\infty$ ). Consider a point  $x_0$  with  $f(x_0) = 0$ . Then  $x(t) = x_0$  is a time-independent solution of the system. In other words,  $x_0$  is a stationary point of the given dynamical system. The linearization of the system about  $x_0$  is the system  $\dot{y} = Ay$  where the matrix  $A$  has entries  $A_i^j = \partial f^j / \partial x^i$ . If each eigenvalue  $\lambda$  of  $A$  has the property that the real part of  $\lambda$  is non-zero then the stationary point is called hyperbolic. In that case the theorem of Hartman and Grobman ([7], p. 120) shows that the linearized system is topologically equivalent to the original system in a neighbourhood of  $x_0$ . In other words, there is a homeomorphism of a neighbourhood of  $x_0$  onto a neighbourhood of the origin in  $\mathbf{R}^n$  such that solutions of the non-linear system are mapped onto solutions of the linearized system. Note that this homeomorphism cannot in general be improved to a  $C^2$  diffeomorphism. (See [7], p. 127.)

When the matrix  $A$  has eigenvalues which are purely imaginary and, in particular, when it has zero as an eigenvalue things are more complicated than for hyperbolic stationary points. Recall that the generalized eigenspace associated to an eigenvalue  $\lambda$  of  $A$  consists of all complex vectors  $x$  such that  $(A - \lambda I)^k x = 0$  for some positive integer  $k$ . The centre subspace  $E_c$  of the dynamical system at  $x_0$  is defined as follows. Take the direct sum of the generalized eigenspaces corresponding to all purely imaginary eigenvalues and then restrict to the real subspace. Then  $E_c$  is an invariant subspace of  $A$  and the restriction of  $A$  to  $E_c$  has purely imaginary eigenvalues.  $E_c$  is the maximal subspace having this property. The centre manifold theorem [3] says that there is a manifold  $M_c$  passing through  $x_0$  which is invariant under the flow of differential equation and whose tangent space at  $x_0$  is  $E_c$ . The manifold  $M_c$  is called a centre manifold. It can be chosen to be  $C^k$  for any positive integer  $k$  but not necessarily to be  $C^\infty$ . (See the example on p. 29 of [3].) It is also in general not unique. However any two centre manifolds of class  $C^k$  are tangent to order  $k$  at  $x_0$  and in fact the coefficients in the Taylor expansion of this submanifold at  $x_0$  can be calculated recursively. Examples of this procedure will be seen later on.

There is a generalization of the Hartman-Grobman theorem to non-hyperbolic stationary points called the reduction theorem [5] which reduces the study of the qualitative behaviour of solutions of a system near a stationary point to the study of solutions on the centre manifold. In favourable cases the qualitative behaviour of solutions on the centre manifold near the origin can be determined once a finite number of expansion coefficients of the centre manifold are known. This will be illustrated by the examples of cosmological models treated later.

The reduction theorem by itself can provide useful information about the qualitative behaviour of solutions of a system of ordinary differential equations. For example this was done for the Mixmaster solution in [8]. (Note, however, that since then a much deeper analysis of this system has been carried out in [11] and [12].) This kind of straightforward application of the reduction theorem is not what is meant by ‘centre manifold theory’ in the present paper.

The latter should rather be reserved for the procedure where the existence of a centre manifold is used as a basis for doing calculations which give concrete information on the nature of solutions. It is calculations of this type which are the main tool of this paper.

Centre manifold theory has the advantage of reducing the dimension of the dynamical system whose qualitative behaviour has to be determined. The dimension of the centre manifold is of course just equal to the number of purely imaginary eigenvalues of the linearization, counting multiplicity. When the centre manifold is one-dimensional things become particularly easy. Determining the stability of the origin in the centre manifold reduces to algebra. This happens in the example studied in Section 6. When the centre manifold is two-dimensional things are a lot more complicated but there is still a far-reaching theory available (see [7], section 2.11) and in particular cases such as those studied in Sections 4 and 5 of this paper, direct approaches may lead to the desired result.

### 3 The Bianchi III equations with massive particles

The results of this section extend the theorems of [10] and the fundamental equations will be taken directly from that source. Here it will be assumed from the start that the Bianchi type is III. The spacetime metric is of the form

$$ds^2 = -dt^2 + a^2(t)(\theta^1)^2 + b^2(t)((\theta^2)^2 + (\theta^3)^2) , \quad (1)$$

where the  $\theta^i$  are one-forms whose exterior derivatives satisfy the relations required to ensure that the metric defines a spacetime of Bianchi type III. For instance we can assume the relations  $d\theta^1 = d\theta^2 = 0$  and  $d\theta^3 = \theta^2 \wedge \theta^3$ . This metric is locally rotationally symmetric (LRS). The matter content of spacetime is described by the spatially homogeneous phase space density of particles  $f(t, v_i)$ . It is assumed that  $f$  is invariant under rotations in the  $(v_2, v_3)$ -plane and the reflection  $v_1 \mapsto -v_1$ . The energy-momentum tensor  $T_{ij}$  for the Einstein-Vlasov system with particles of mass  $m \geq 0$  is diagonal and is described by

$$\begin{aligned} \rho &= \int f_0(v_i)(m^2 + a^{-2}(v_1)^2 + b^{-2}((v_2)^2 + (v_3)^2))^{1/2}(ab^2)^{-1} dv_1 dv_2 dv_3 , \\ p_i &= \int f_0(v_i) g^{ii}(v_i)^2 (m^2 + a^{-2}(v_1)^2 + b^{-2}((v_2)^2 + (v_3)^2))^{-1/2} (ab^2)^{-1} dv_1 dv_2 dv_3 \end{aligned} \quad (2)$$

where  $\rho$  is the energy density and  $p_i = T^i_i$  the pressure components of the energy-momentum tensor. The function  $f_0$  is determined at some fixed time  $t_0$  by  $f_0(v_i) = f(t_0, v_i)$  where  $f$  is the phase space density of particles. Some further technical conditions will be imposed on  $f_0$ . It is assumed to be non-negative and have compact support. It is also assumed that the support does not intersect the coordinate planes  $v_i = 0$ . A function  $f_0$  with this property will be said to have split support. This assumption ensures that the dynamical

system which describes the evolution of the spacetimes of interest has smooth coefficients.

The momentum constraints are automatically satisfied for these models. Only the Hamiltonian constraint and the evolution equations are left. Instead of considering a set of second order equations in terms of  $a$  and  $b$ , we will reformulate these equations as a first order system of ODEs by introducing a new set of variables. The mean curvature  $\text{tr}k$  (where  $k_{ij}$  is the second fundamental form) is given by

$$\text{tr}k = -(a^{-1}da/dt + 2b^{-1}db/dt) . \quad (3)$$

A new dimensionless time coordinate  $\tau$  is defined by  $-\frac{1}{3} \int_{t_0}^t \text{tr}k(t)dt$  for some arbitrary fixed time  $t_0$ . In the following a dot over a quantity denotes its derivative with respect to  $\tau$ . The Hubble variable  $H$  is given by  $H = -\text{tr}k/3$ . This section and the next are concerned with the case of massive particles ( $m > 0$ ). Define the following dimensionless variables:

$$\begin{aligned} z &= m^2/(a^{-2} + 2b^{-2} + m^2) , \\ s &= b^2/(b^2 + 2a^2) , \\ M_3 &= 3b^{-2}(\text{tr}k)^{-2} , \\ \Sigma_+ &= -3(b^{-1}db/dt)(\text{tr}k)^{-1} - 1 , \end{aligned} \quad (4)$$

These variables lead to a decoupling of the equation for the only remaining dimensional variable  $H$  (or equivalently  $\text{tr}k$ )

$$\dot{H} = -(1 + q)H , \quad (5)$$

where the deceleration parameter  $q$  is given by

$$q = 2\Sigma_+^2 + \frac{1}{2}\Omega(1 + R) . \quad (6)$$

The quantity  $R$  is defined by

$$R = (p_1 + 2p_2)/\rho , \quad (7)$$

where

$$\begin{aligned} p_1/\rho &= (1 - z)sg_1/h , \\ p_2/\rho &= \frac{1}{2}(1 - z)(1 - s)g_2/h , \\ g_{1,2} &= \int f_0(v_i)(v_{1,2})^2 [z + (1 - z)(s(v_1)^2 + \frac{1}{2}(1 - s)((v_2)^2 + (v_3)^2))]^{-1/2} dv_1 dv_2 dv_3 , \\ h &= \int f_0(v_i)[z + (1 - z)(s(v_1)^2 + \frac{1}{2}(1 - s)((v_2)^2 + (v_3)^2))]^{1/2} dv_1 dv_2 dv_3 . \end{aligned} \quad (8)$$

The assumption of split support ensures that the function  $R(s, z)$  is a smooth ( $C^\infty$ ) function of its arguments. The related quantity  $R_+$  defined by

$$R_+ = (p_2 - p_1)/\rho \quad (9)$$

is a smooth function of  $s$  and  $z$  for the same reason.

The normalized energy density  $\Omega = \rho/(3H^2)$  is determined by the Hamiltonian constraint and, in units where  $G = 1/8\pi$ , is given by

$$\Omega = 1 - \Sigma_+^2 - M_3 . \quad (10)$$

The assumption of a distribution of massive particles with non-negative mass leads to inequalities for  $R$ ,  $R_+$  and  $\Omega$ . Firstly,  $0 \leq R \leq 1$  with  $R = 0$  only when  $z = 1$  and  $R = 1$  only when  $z = 0$ . Secondly,  $-R \leq R_+ \leq \frac{1}{2}R$  with  $R_+ = \frac{1}{2}R$  for  $s = 0$  and  $R_+ = -R$  for  $s = 1$ . Thirdly  $\Omega \geq 0$ . Using these inequalities in equation (6) in turn results in  $0 \leq q \leq 2$  (i.e., the same inequality as for causal perfect fluids, see [13]).

The remaining dimensionless coupled system is:

$$\begin{aligned} \dot{\Sigma}_+ &= -(2 - q)\Sigma_+ + M_3 + \Omega R_+ , \\ \dot{s} &= 6s(1 - s)\Sigma_+ , \\ \dot{z} &= 2z(1 - z)(1 + \Sigma_+ - 3\Sigma_+s) , \\ \dot{M}_3 &= 2(q - \Sigma_+)M_3 , \end{aligned} \quad (11)$$

In the following this is referred to for short as the Bianchi III system. It is of interest to note that the metric functions  $a, b$  are expressible in terms of  $s, z$  in the massive case. The relations are

$$a^2 = z(m^2s(1 - z))^{-1}, \quad b^2 = 2z(m^2(1 - s)(1 - z))^{-1} . \quad (12)$$

There are a number of boundary submanifolds:

$$\begin{aligned} z &= 0, 1 , \\ s &= 0, 1 , \\ \Omega &= 0 . \end{aligned} \quad (13)$$

The submanifold  $z = 0$  corresponds to the massless case. The submanifold  $z = 1$  leads to a decoupling of the  $s$ -equation, leaving a system identical to the corresponding dust equations. The submanifolds  $s = 0, s = 1$  correspond to problems with  $f_0$  being a distribution while  $\Omega = 0$  constitutes the vacuum submanifold with test matter.

Including these boundaries yields a compact state space. In order to apply the standard theory of dynamical systems the coefficients must be  $C^1$  on the entire compact state space  $G$  of a given model. This is necessary even for uniqueness. In the present case it suffices to show that  $R$  and  $R_+$  are  $C^1$  on  $G$ , i.e., that they are  $C^1$  for  $s, z$  when  $0 \leq s \leq 1, 0 \leq z \leq 1$ . As has already been pointed out, this follows from the assumption of split support, which even implies the analogous statement with  $C^1$  replaced by  $C^\infty$ . It would be possible to get  $C^1$  regularity under the weaker assumption that  $f_0$  vanishes as fast as a sufficiently high power of the distance to the coordinate planes but this is of little relevance to the main concerns of this paper.

Of key importance is the existence of a monotone function in the ‘massive’ interior part of the state space:

$$\begin{aligned} M &= (s(1-s)^2)^{-1/3} z(1-z)^{-1} , \\ \dot{M} &= 2M . \end{aligned} \tag{14}$$

Note that the volume  $ab^2$  is proportional to  $M^{3/2}$ . This monotone function rules out any interior  $\omega$ - and  $\alpha$ -limit sets and forces these sets to lie on the  $s = 0$ ,  $s = 1$ ,  $z = 0$  or  $z = 1$  parts of the boundary.

The physical state space,  $G$ , of the LRS type III models is given by the region in  $\mathbf{R}^4$  defined by the inequalities  $M_3 \geq 0$ ,  $0 \leq s \leq 1$ ,  $0 \leq z \leq 1$ , and  $1 - \Sigma_+^2 - M_3 \geq 0$ . A solution of the Bianchi III system (11) will be said to lie in the physical region if it lies in the interior of  $G$ . In the following we will be interested in the behaviour in the limit  $\tau \rightarrow \infty$  of the solutions of the Bianchi III equations which lie in the physical region. This corresponds to the asymptotics of the underlying cosmological model in a phase of unlimited expansion. It was shown in [10] that as  $\tau$  tends to infinity a solution in the physical region converges to a point of the line of stationary points  $L_3$  which consists of all points of the form  $(\frac{1}{2}, 1, z_0, \frac{3}{4})$ . The point to which the solution converges must satisfy  $z_0 > 0$ . It was left open in [10] whether that point must satisfy  $z_0 = 1$  or whether values less than one are also possible. In the next section it will be shown that in fact  $z \rightarrow 1$  as  $\tau \rightarrow \infty$ .

## 4 Centre manifold analysis for the Bianchi III equations with massive particles

It follows from Theorem 5.1 of [10] that any solution of the Bianchi III system in the physical region converges to a limit of the form  $(\frac{1}{2}, 1, z_0, \frac{3}{4})$  with  $0 < z_0 \leq 1$  as  $\tau \rightarrow \infty$  and that no solution in the physical region converges to a limit of this form as  $\tau \rightarrow -\infty$ . In order to decide which values of  $z_0$  can occur as limiting values of solutions in the physical region in the forward time direction it is necessary to study the behaviour of solutions close to the stationary points on the line  $L_3$  in detail. If the system is linearized about one of these points then it is seen that the linearization has two negative eigenvalues and two zero eigenvalues. The fact that there is a whole line of stationary points implies that there is automatically one zero eigenvalue. However the second zero eigenvalue means that there is an extra degeneracy. For  $z_0 = 1$  the matrix defining the linearized system is diagonalizable while for  $0 < z_0 < 1$  it is not. For this reason there are two significantly different cases to be considered.

Consider first the case  $z_0 = 1$ . New coordinates will be introduced in order to simplify the analysis. First the stationary point with  $z_0 = 1$  will be translated to the origin. Define  $x = \Sigma_+ - \frac{1}{2}$ ,  $y = 1 - s$ ,  $w = M_3 - \frac{3}{4}$ ,  $\tilde{z} = z - 1$ . Then the transformed system is

$$\dot{x} = -\frac{1}{4} + q(\frac{1}{2} + x) - 2x + w + \Omega R_+ \tag{15}$$

$$\dot{y} = -6y(1-y)(\frac{1}{2} + x) \quad (16)$$

$$\dot{\tilde{z}} = -2(1 + \tilde{z})\tilde{z}(-2x + \frac{3}{2}y + 3xy) \quad (17)$$

$$\dot{w} = 2(q - x - \frac{1}{2})(w + \frac{3}{4}) \quad (18)$$

$$q = \frac{1}{2} + 2x + 2x^2 + \frac{1}{2}\Omega(1 + R) \quad (19)$$

$$\Omega = -x - w - x^2 \quad (20)$$

Next set  $Z = -\tilde{z}$ ,  $u = x + w$  and  $v = x - w$ . In order to determine the behaviour of solutions of the resulting system near the origin we attempt to use the linearization at that point. The linearization has zero as an eigenvalue with a corresponding two-dimensional eigenspace. This eigenspace is the centre subspace. Centre manifold theory [3] tells us that there is an invariant manifold through the origin whose tangent space is the centre subspace. This manifold is a centre manifold. A centre manifold does not in general need to be unique. However approximations to it can be derived and these often suffice to determine the qualitative behaviour of solutions on the centre manifold. The qualitative behaviour of solutions near the stationary point is determined by the qualitative behaviour of solutions on any centre manifold. A general property of centre manifolds is that a stationary point has an open neighbourhood such that any other stationary point in that neighbourhood lies on any centre manifold of the original point. It follows that in a neighbourhood of the stationary point of the system under consideration the part of the line  $L_3$  close to that point lies on any centre manifold. In the case of present interest the centre manifold can be defined by the equations  $y = \phi(u, Z)$  and  $v = \psi(u, Z)$  where  $\phi$  and  $\psi$  vanish at least quadratically at the origin. The invariance of the centre manifold implies that

$$\dot{y} = (\partial\phi/\partial u)\dot{u} + (\partial\phi/\partial Z)\dot{Z} \quad (21)$$

$$\dot{v} = (\partial\psi/\partial u)\dot{u} + (\partial\psi/\partial Z)\dot{Z} \quad (22)$$

on the centre manifold. The right hand sides of these equations vanish to at least third order at the origin. On the centre manifold

$$\dot{y} = -3\phi - 3\phi(u + \psi) + 3\phi^2(1 + u + \psi) \quad (23)$$

Substituting (23) into (21) shows that  $\phi$  vanishes at quadratic order. Putting this information back into the equation (21) shows that  $\phi$  vanishes at third order. This can be repeated indefinitely, with the result that  $\phi$  vanishes to all orders. On the centre manifold

$$\begin{aligned} \dot{v} = & -\frac{3}{2}\psi - \frac{3}{8}u^2 + \frac{1}{4}u\psi + \frac{5}{8}\psi^2 - \frac{1}{4}u^3 + \frac{1}{4}u^2\psi + u\psi^2 + \frac{3}{4}\psi^3 \\ & + \Omega(R_+ - \frac{1}{2}R) + \frac{1}{4}\Omega(1 + R)(3\psi - u) \end{aligned} \quad (24)$$

Note that  $(R + R_+)(y, Z) = O(y)$  since  $R + R_+$  is smooth and vanishes when  $y = 0$ . Since  $\phi$  vanishes to all orders it follows that  $R_+$  and  $-R$  are equal to all orders on the centre manifold. Hence to quadratic order  $\psi(u, Z) = -\frac{1}{12}u^2 - r_2uZ + \dots$



where  $r_2$  is the derivative of  $R$  with respect to  $z$ , evaluated at the stationary point of interest. Next the dynamics on the centre manifold will be examined.

$$\begin{aligned} \dot{u} = & u^2 + \frac{11}{4}u\psi + \frac{3}{4}\psi^2 + \frac{9}{16}u^3 + \frac{15}{16}u^2\psi + \frac{3}{16}u\psi^2 - \frac{3}{16}\psi^3 \\ & + \Omega[(R + R_+) + \frac{1}{4}R(3u - \psi)] \end{aligned} \quad (25)$$

while

$$\dot{Z} = 2uZ - 2uZ^2 + 2Z\psi - 2Z^2\psi - 3Z(1 - Z)\phi(1 + u + \psi) \quad (26)$$

Discarding terms of order greater than two gives the truncated system

$$\dot{u} = u^2 \quad (27)$$

$$\dot{Z} = 2uZ \quad (28)$$

For this system  $u^{-2}Z$  is a conserved quantity and so the qualitative behaviour of the solutions is easily determined. Is the qualitative behaviour of solutions of the full system on the centre manifold similar? The terms on the right hand side of the restriction of the evolution equations to the centre manifold written out explicitly above contain a factor of  $u$ . This is not an accident and in fact the full equations are of the form

$$\dot{u} = uf(u, Z) \quad (29)$$

$$\dot{Z} = ug(u, Z) \quad (30)$$

for some functions  $f$  and  $g$  of any arbitrary finite degree of differentiability. This is because of the fact, already mentioned above, that the line  $L_3$  of stationary points is locally contained in the centre manifold. Thus the restriction of the system to the centre manifold has a line of stationary points with  $u = 0$ . This means that the vector field defining the dynamical system vanishes for  $u = 0$ . The existence of the functions  $f$  and  $g$  then follows from Taylor's theorem. The fact that  $L_3$  lies on the centre manifold also implies that  $\psi(u, Z) = uh(u, Z)$  for a function  $h$  of arbitrary finite differentiability.

It is possible to get more precise information about the function  $f$  as follows. Recall that  $R + R_+$  vanishes modulo the function  $\phi$ . Since  $\phi$  vanishes to all orders it follows that expressions containing a factor  $R + R_+$  can be ignored in all perturbative calculations. We have

$$\Omega = -u - \frac{1}{4}u^2 - \frac{1}{2}u\psi - \frac{1}{4}\psi^2 \quad (31)$$

It follows that every term in the evolution equation for  $u$  on the centre manifold contains a factor of  $u^2$ , either directly or via the fact that  $\psi = uh$ . Hence  $f(0, Z) = 0$ .

For the purpose of qualitative analysis we can replace the above system by the system

$$u' = f(u, Z) + \dots \quad (32)$$

$$Z' = g(u, Z) + \dots \quad (33)$$

which has the same integral curves away from the  $Z$ -axis. Only solutions with  $Z \geq 0$  are physical. The  $Z$ -axis is an invariant manifold of the rescaled dynamical system. This new system has a hyperbolic stationary point at the origin which is a source. This makes it easy to determine the phase portrait for the system near the endpoint of  $L_3$  at  $z = 1$ . What we see is that there is an open set of initial data in the physical region such that the corresponding solutions approach the stationary point with  $z = 1$  as  $\tau \rightarrow \infty$  and there is a neighbourhood of the endpoint of  $L_3$  at  $z = 1$  such that in that neighbourhood no solution in the physical region approaches a point of  $L_3$  with  $z < 1$ .

Consider now the case  $0 < z_0 < 1$ . The analysis is similar to that in the case already considered, although the calculations are somewhat heavier. For  $z_0 < 1$  the linearization has a non-diagonal Jordan form. Define  $x = \Sigma_+ - \frac{1}{2}$ ,  $y = 1 - s$ ,  $w = M_3 - \frac{3}{4}$ ,  $\tilde{z} = z - z_0$ . Then the transformed system is

$$\dot{x} = -\frac{1}{4} + q(\frac{1}{2} + x) - 2x + w + \Omega R_+ \quad (34)$$

$$\dot{y} = -6y(1 - y)(\frac{1}{2} + x) \quad (35)$$

$$\dot{\tilde{z}} = 2(z_0 + \tilde{z})(1 - z_0 - \tilde{z})(-2x + \frac{3}{2}y + 3xy) \quad (36)$$

$$\dot{w} = 2(q - x - \frac{1}{2})(w + \frac{3}{4}) \quad (37)$$

$$q = \frac{1}{2} + 2x + 2x^2 + \frac{1}{2}\Omega(1 + R) \quad (38)$$

$$\Omega = -x - w - x^2 \quad (39)$$

The linearization has zero as an eigenvalue with a corresponding two-dimensional generalized eigenspace.

For calculational purposes it is convenient to choose coordinates which reduce the linearization to a simple form. For this reason the following variables will be defined. They almost reduce the linearized operator to Jordan form.

$$U = -2z_0(1 - z_0)(x + w) \quad (40)$$

$$V = -\frac{1}{2}[(\bar{R} - 1)x + (\bar{R} + 1)w] \quad (41)$$

$$Y = y \quad (42)$$

$$Z = \tilde{z} + z_0(1 - z_0)y + \frac{4}{3}[(\bar{R} - 1)x + (\bar{R} + 1)w]z_0(1 - z_0) \quad (43)$$

Here  $\bar{R}$  is the value of  $R$  when  $s = 1$  and  $z = z_0$ . Note that the value of  $R_+$  when  $s = 1$  and  $z = z_0$  is  $-\bar{R}$ . Up to linear order the equations in the new coordinates are

$$\dot{U} = 0 + \dots \quad (44)$$

$$\dot{V} = -\frac{3}{2}V + \dots \quad (45)$$

$$\dot{Y} = -3Y + \dots \quad (46)$$

$$\dot{Z} = (\bar{R} + 1)U + \dots \quad (47)$$

In the new coordinates the centre subspace is defined by  $V = 0$  and  $Y = 0$ . The centre manifold is defined by equations of the form  $Y = \phi(U, Z)$  and  $V = \psi(U, Z)$ , where  $\phi$  and  $\psi$  are functions of any desired finite degree of differentiability which vanish at least quadratically at the origin. (These are not

the same functions  $\phi$  and  $\psi$  introduced in the centre manifold calculation for  $z_0 = 1$ .) Now apply the equations

$$\dot{Y} = (\partial\phi/\partial U)\dot{U} + (\partial\phi/\partial Z)\dot{Z} \quad (48)$$

$$\dot{V} = (\partial\psi/\partial U)\dot{U} + (\partial\psi/\partial Z)\dot{Z} \quad (49)$$

which must hold on the centre manifold. The linear contribution to  $\dot{Z}$  is proportional to  $U$  and hence the quadratic contribution to the right hand side of the first equation contains only terms proportional to  $U^2$  and  $UZ$  and no term proportional to  $Z^2$ . The quadratic part of the left hand side is proportional to the quadratic part of  $\phi$ . Suppose that the latter is given by  $AU^2 + BUZ + CZ^2$  for constants  $A$ ,  $B$  and  $C$ . Comparing the coefficients of  $Z^2$  shows that  $C = 0$ . Then comparing coefficients of  $UZ$  shows that  $B = 0$ . At this point it has been shown that the right hand side of the equation vanishes to quadratic order and it follows that  $A = 0$ . Hence  $\phi$  vanishes to quadratic order. As in the case  $z_0 = 1$  this information can be used repeatedly to show that  $\phi$  vanishes to all orders at the origin.

Next the equation for  $\dot{V}$  on the centre manifold will be examined. It turns out that the quadratic contribution to  $\dot{V}$  is a linear combination of  $U^2$  and  $UZ$ . It can be concluded that up to quadratic order  $\psi = \alpha U^2 + \beta UZ + \dots$  for some constants  $\alpha$  and  $\beta$ . These constants depend on the value of  $z_0$  at the point where the linearization is carried out and on the initial data  $f_0$ .

Substituting the information about the centre manifold into the original system shows that the restriction of the dynamical system to the centre manifold has the following form up to quadratic order:

$$\dot{U} = \gamma U^2 + \dots \quad (50)$$

$$\dot{Z} = (\bar{R} + 1)U + \delta U^2 + \epsilon UZ + \dots \quad (51)$$

for some constants  $\gamma$ ,  $\delta$  and  $\epsilon$ .

All the terms on the right hand side of the restriction of the evolution equations to the centre manifold written out explicitly above contain a factor of  $U$  and for the same reason as in the case  $z_0 = 1$  the full equations are of the form

$$\dot{U} = Uf(U, Z) \quad (52)$$

$$\dot{Z} = Ug(U, Z) \quad (53)$$

for some functions  $f$  and  $g$  of arbitrary finite differentiability. Thus the restriction of the system to the centre manifold has a line of stationary points with  $U = 0$ .

Consider now the dynamical system defined by

$$U' = f(U, Z) \quad (54)$$

$$Z' = g(U, Z) \quad (55)$$

where the prime denotes  $d/d\sigma$  for a time coordinate  $\sigma$ . This has the same integral curves as the system we want to analyse in the complement of the  $Z$ -axis. It will be referred to in the following as the rescaled system. At the origin

$f$  vanishes while  $g$  has the non-vanishing value  $\bar{R} + 1$ . In fact  $f(0, Z) = 0$  for all  $Z$ , the argument being just as in the case  $z_0 = 1$ . As a consequence the integral curve of the rescaled system passing through the origin lies on the  $Z$ -axis. Since the vector field defining this system is non-vanishing at the origin it is clear that no solution can approach the origin from the physical region.

Putting together the results of the two centre manifold analyses with Theorem 5.1 of [10] yields the following theorem:

**Theorem 1** If a smooth non-vacuum reflection-symmetric LRS solution of Bianchi type III of the Einstein-Vlasov equations for massive particles is represented as a solution of (11) then for  $\tau \rightarrow \infty$  it converges to the point with coordinates  $(\frac{1}{2}, 1, 1, \frac{3}{4})$ . The quantities  $p_1/\rho$  and  $p_2/\rho$  converge to zero as  $t \rightarrow \infty$ .

Next some information will be obtained on the detailed asymptotic behaviour of the spacetime as  $t \rightarrow \infty$ . Consider first the behaviour of a solution of the rescaled dynamical system for  $z_0 = 1$  as  $\sigma \rightarrow -\infty$ . At the origin the rescaled system has a stationary point which is a hyperbolic source. By a theorem of Hartman (see [7], p. 127) the system can be linearized by a  $C^1$  mapping. As a consequence  $u = u_0 e^\sigma + o(e^\sigma)$  for some constant  $u_0$  and  $Z = o(e^\sigma)$ . Along a solution of the equations  $d\tau/d\sigma = u^{-1}(\sigma)$ . Hence for large negative values of  $\sigma$  we have  $\tau = -u_0^{-1} e^{-\sigma} + o(e^{-\sigma})$ . Hence on the centre manifold  $u = -\tau^{-1} + o(\tau^{-1})$  and  $Z = o(\tau^{-1})$ .

General results on centre manifolds [3] show that the general solution satisfies

$$u(\tau) = -\tau^{-1} + o(\tau^{-1}) \quad (56)$$

$$z(\tau) = 1 + o(\tau^{-1}) \quad (57)$$

$$y(\tau) = o(\tau^{-k}) \quad (58)$$

$$v(\tau) = -\frac{1}{12}\tau^{-2} + o(\tau^{-2}) \quad (59)$$

where  $k$  is any positive integer. As  $\tau \rightarrow \infty$  we have  $q = \frac{1}{2} + o(1)$ . Now  $dH/dt = -(1+q)H^2$ . It follows that  $H = \frac{2}{3}t^{-1}(1+o(1))$ . Thus in leading order  $\tau = \frac{2}{3}\log t + \dots$ . Putting this information in the asymptotic expressions above gives the leading order behaviour of  $u$ ,  $z$ ,  $y$  and  $v$  as functions of  $t$ . This in turn gives the asymptotic behaviour of the variables  $\Sigma_+$ ,  $s$ ,  $M_3$  and  $z$ . The result is

$$\Sigma_+(t) = \frac{1}{2} - \frac{3}{4}(\log t)^{-1} + \dots \quad (60)$$

$$s = 1 + o((\log t)^{-k}) \quad (61)$$

$$M_3 = \frac{3}{4} - \frac{3}{4}(\log t)^{-1} + \dots \quad (62)$$

$$z = 1 + o((\log t)^{-1}) \quad (63)$$

From this it can be concluded using (12) that  $a/(\log t)^{1/2}$  goes to infinity as  $t \rightarrow \infty$ . It follows directly from the fundamental equations that  $a$  goes to infinity slower than any power of  $t$ . To see this note that since  $\Sigma_+$  tends to  $\frac{1}{2}$ , the quantity  $(b^{-1}db/dt)(\text{tr}k)^{-1}$  tends to  $-\frac{1}{2}$ . Hence  $b^{-1}db/dt = t^{-1} + \dots$  and  $a^{-1}da/dt = o(t^{-1})$ . This gives the desired conclusion concerning the upper bound for the rate of growth of  $a$ . The leading order behaviour of  $b$  can be read off from the defining equation for  $M_3$ , with the result that  $b = t + \dots$

## 5 Centre manifold analysis for the Bianchi III equations with massless particles

The analysis of LRS Bianchi spacetimes with massless particles in [9] included the case of Bianchi type III. In this section the results obtained in [9] will be sharpened. As previously indicated, the dynamical system for LRS Bianchi type III spacetimes with massless particles is equivalent to the boundary component  $z = 0$  of the region  $G$  on which the system for massive particles is defined. Here we adopt the notation of [10] rather than that of [9]. Then the dynamical system required in this section can be obtained by specializing a system we had in the last section. The result is

$$\dot{x} = -\frac{1}{4} + q(\frac{1}{2} + x) - 2x + w + \Omega R_+ \quad (64)$$

$$\dot{y} = -6y(1 - y)(\frac{1}{2} + x) \quad (65)$$

$$\dot{w} = 2(q - x - \frac{1}{2})(w + \frac{3}{4}) \quad (66)$$

$$q = \frac{1}{2} + 2x + 2x^2 + \Omega \quad (67)$$

$$\Omega = -x - w - x^2 \quad (68)$$

The only modifications are that the equation for  $z$  has been dropped and that the relation  $R = 1$  has been used. This is a three-dimensional dynamical system. Its linearization at the origin has eigenvalues 0,  $-\frac{3}{2}$  and  $-3$  and its kernel is spanned by  $(1, 0, 0)$ . It follows that this point has a one-dimensional centre manifold. It was already shown in [9] that all solutions of this system corresponding to solutions of the Einstein-Vlasov equations with massless particles converge to the origin as  $\tau \rightarrow \infty$ . The purpose of the centre manifold analysis here is to obtain more detailed information about how this point is approached and hence how the scale factors behave in the phase of unlimited expansion. It is useful to substitute in the expressions for  $\Omega$  and  $q$ , thus obtaining the following dynamical system:

$$\dot{x} = \frac{3}{2}w + \frac{5}{2}x^2 - xw + x^3 - (x + w + x^2)(1 + R_+) \quad (69)$$

$$\dot{y} = -6y(1 - y)(\frac{1}{2} + x) \quad (70)$$

$$\dot{w} = 2(-w + x^2)(w + \frac{3}{4}) \quad (71)$$

The centre manifold is defined by the equations  $y = \phi(x)$  and  $w = \psi(x)$ . Using the same procedure as in the last section the functions  $\phi$  and  $\psi$  can be determined up to quadratic order. In this case it turns out that  $\phi$  vanishes to quadratic order while to that order  $\psi = x^2 + \dots$ . Substituting this information into the evolution equation for  $x$  shows that  $\dot{x} = 4x^2 + O(|x|^3)$  on the centre manifold. Since physical solutions converge to the origin as  $\tau \rightarrow \infty$  they must be approximated by solutions on the centre manifold with  $x$  negative. It can be concluded that for these solutions  $\Sigma_+$  is eventually less than one half.

With the information already obtained, various features of the dynamics of the spacetime can be reconstructed. To leading order  $x = -(1/4\tau) + \dots$ ,  $y = o(\tau^{-2})$  and  $w = 1/16\tau^2 + o(\tau^{-2})$ . The quantity  $q$  tends to  $\frac{1}{2}$  and so some

of the calculations for the massive case can be taken over here. As in that case  $b = t + \dots$ . It is not possible to determine the asymptotic behaviour of the scale factor  $a$  in the same way as in the massive case; a different approach is necessary. It may be computed that  $a^{-1}da/dt = \frac{2}{3}(\text{tr}k)x$ . Hence in leading order  $a^{-1}da/dt = \frac{1}{2}(t \log t)^{-1} + \dots$ . It follows that  $a \rightarrow \infty$  as  $t \rightarrow \infty$ .

## 6 Inflation with a massive scalar field

This section presents a rigorous derivation of some heuristic results of [2] for spatially flat homogeneous and isotropic spacetimes with a massive scalar field as source. The starting point is the following system of equations from [2].

$$x_\eta = y \quad (72)$$

$$y_\eta = -x - 3y(x^2 + y^2)^{1/2} \quad (73)$$

This is a dynamical system on the plane whose coefficients are smooth everywhere except at the origin, where they are  $C^1$ . Transforming to polar coordinates  $(r, \theta)$  and introducing  $\rho = r/(1 + r)$  gives the system

$$\rho_\eta = -3\rho^2 \sin^2 \theta \quad (74)$$

$$\theta_\eta = -1 - 3\rho(1 - \rho)^{-1} \sin \theta \cos \theta \quad (75)$$

Introducing a new time coordinate  $\tau$  and setting  $u = 1 - \rho$  gives

$$u_\tau = 3u(1 - u)^2 \sin^2 \theta \quad (76)$$

$$\theta_\tau = -u - 3(1 - u) \sin \theta \cos \theta \quad (77)$$

So far this is nothing new compared to what is done in [2]. Now the stationary point at the origin will be investigated. The eigenvalues of the linearization are  $-3$  and zero. Evidently the  $\theta$ -axis is invariant and in fact it is the stable manifold of the origin. The centre subspace is given by  $u + 3\theta = 0$  and thus we introduce  $v = u + 3\theta$  in order to study the centre manifold. This leads to the transformed system

$$u_\tau = 3u(1 - u)^2 \sin^2(\frac{1}{3}(v - u)) \quad (78)$$

$$\begin{aligned} v_\tau = & -3u \cos^2(\frac{1}{3}(v - u)) - 6u^2 \sin^2(\frac{1}{3}(v - u)) + 3u^3 \sin^2(\frac{1}{3}(v - u)) \\ & - 9 \sin(\frac{1}{3}(v - u)) \cos(\frac{1}{3}(v - u)) + 9u \sin(\frac{1}{3}(v - u)) \cos(\frac{1}{3}(v - u)) \end{aligned} \quad (79)$$

The centre manifold is of the form  $v = \phi(u)$ . To quadratic order  $\phi(u) = -u^2 + \dots$ . On the centre manifold  $u_\tau = \frac{1}{3}u^3 + O(u^4)$  and hence there is a unique solution which enters the physical region.

The asymptotic form of the solution as  $\tau \rightarrow -\infty$  is  $u = \sqrt{3/2}(-\tau)^{-1/2} + \dots$ . It follows that  $v = \frac{3}{2}\tau^{-1} + \dots$ . Hence

$$\rho = 1 - \sqrt{3/2}(-\tau)^{-1/2} + \dots \quad (80)$$

and

$$\theta = -\sqrt{1/6}(-\tau)^{-1/2} + \dots \quad (81)$$

The next step is to convert back to the original variables. In leading order  $\eta = -\sqrt{6}(-\tau)^{1/2} + \dots$  and so  $\rho = 1 + 3/\eta + \dots$  and  $\theta = 1/6\eta + \dots$ . It follows that  $x = \eta/3 + \dots$  and  $y = \frac{1}{18} + \dots$ . The combination  $z = \sqrt{x^2 + y^2}$  occurring in [2] is asymptotic to  $\eta/3$ . The linear dependence of  $x$  and  $z$  with respect to  $\eta$  is what was found in [2]. The interpretation of these variables is that  $\eta$ ,  $x$  and  $z$  are proportional to proper time, the scalar field  $\phi$  and the mean curvature of the hypersurfaces of constant time respectively. Thus in the limit  $t \rightarrow -\infty$  both  $\phi$  and the mean curvature are proportional to  $t$ . The leading order behaviour of the scale factor for large negative times is, up to inessential constants,  $e^{t^2}$ . This gives a rigorous confirmation of the conclusions of [2]. Note that the relevance of centre manifolds in this context has been pointed out by Foster [4] but that he did not carry out a full centre manifold analysis; his paper was mainly concerned with other applications of dynamical systems to homogeneous and isotropic solutions of the Einstein equations coupled to a scalar field with potential.

## 7 Further comments

This section contains some further comments on the results of Sections 4 and 5. In terms of the dimensionless variables which are the unknowns in the dynamical system studied in those sections, solutions of the equations with collisionless matter (with both massless and massive particles) converge at late times to a point corresponding to a vacuum solution of Bianchi type III. The corresponding statement holds for LRS Bianchi type III solutions with dust. (Cf. the discussion in section 6 of [9].) In particular, the dimensionless density parameter  $\Omega$  tends to zero in the expanding direction. Furthermore, in the case of massive particles, the ratios  $T_i^t/\rho$  of the spatial eigenvalues of the energy-momentum tensor (i.e. the principal pressures) to the energy density tend to zero. Thus in a certain sense both solutions with collisionless matter and dust solutions are approximated by vacuum solutions, while solutions with collisionless matter and massive particles are approximated better by dust solutions than either of these are approximated by vacuum solutions. This is one of the main results of this paper.

One of the scale factors,  $a$ , grows much more slowly than the other scale factor  $b$ . An important conclusion is that  $a$  tends to infinity at all in the case of collisionless matter, both in the massive and massless cases. Centre manifold theory was the essential tool which allowed us to prove this and thus to go beyond what was achieved in [9] and [10]. In this sense both types of collisionless matter resemble each other and differ from the vacuum case, where  $a$  is asymptotically constant. These results lend further support to the following two suggestions made in [9] and [10]. The first is that any spatially homogeneous solution of the Einstein equations with collisionless matter and massive particles behaves asymptotically in a phase of unlimited expansion like a dust

model. The second is that in any non-vacuum spatially homogeneous model with perfect fluid or collisionless matter eventually every scale factor increases in the time direction in which the volume increases. It would be interesting to investigate the truth of these statements in solutions of the Einstein equations with collisionless matter which are LRS Bianchi type VIII or type II but not LRS.

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